SURFACE FRICTION IN THE ADIABATIC FLOW OF A COMPRESSIBLE GAS IN TUBES

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From an analysis of the experimental data [1] we have found the relationship between surface friction and the Mach number.

Below we examine the flow of gas in a tube at a subsonic velocity at the inlet, and with an exhaust crises. All of the quantities are dimensionless. The limit velocity serves as the velocity scale and the stagnation temperature represents the temperature scale; for the pressure and density scales we use, repectively, the pressure and density at the initial section. The coefficients of dynamic and turbulent viscosity have been referred to the dynamic viscosity at the wall. It is further assumed that $y = \bar{y}r_0$, $x = \bar{x}r_0\tilde{Re}$, $X = x/2r_0$, and $\tilde{Re}$ is referred to $r_0/2$, to the velocity scale, and to the viscosity scale.

The basic results [1] can be formulated in the following fashion:

a) in practical terms, the pressure is a function only of the longitudinal coordinate x; it diminishes with increasing speed as it approaches the exhaust, so that $dp/dx \to -\infty$ as $x \to x_f$;

b) the stagnation temperature is kept virtually constant at all points in the flow;

c) the profile of the longitudinal velocity, with approach to the exhaust, fills out all the more rapidly, thickening at the center;

d) this profile, with the exception of small areas near the axis and the wall, is rather well approximated by the exponential relationship $u = u_1y^n$, where the exponent n diminishes very rapidly toward the exhaust, attaining values of the order of 1/15 for $Re_\theta \approx 10^6$;

e) the mean-mass velocity w on approach to the exhaust increases very rapidly, reaching supersonic values at $x = x_f$.

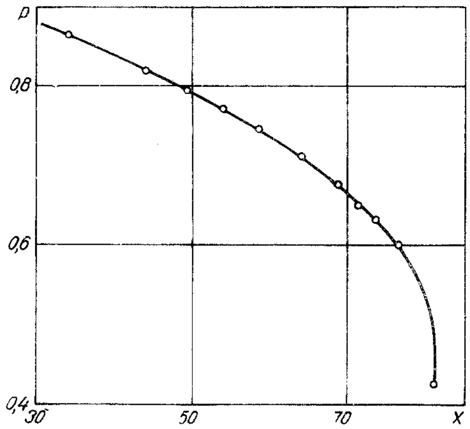

Fig. 1. Pressure p as a function of the dimensionless length X (the points show the experiment described in [1], and the curve represents the approximation according to (15), $\alpha = 0.35$).

At our request, the authors of [1] were kind enough to give us their primary data, which together with certain limit relationships characterizing the crises, enabled us to obtain information as to the nature of the change in the surface friction.

It was assumed throughout in [3] that $\tau_0 < \infty$, all the way to $x = x_f$. At the same time, the nature of the variation in the velocity profile shows that as $x \to x_f$ we must have $\tau_0 \to \infty$. This is not a paradoxical result: it is in agreement with the Falkner and Skan [2] result familiar from the theory of the laminar boundary layer.

To prove our statement, we will examine the radius of curvature for the profile of the longitudinal velocity at the tube wall:

$$r_{f0} = \frac{\left[1 + \left(\frac{\partial u}{\partial y}\right)_0^2\right]^{3/2}}{\left(\frac{\partial^2 u}{\partial y^2}\right)_0}. \quad (1)$$

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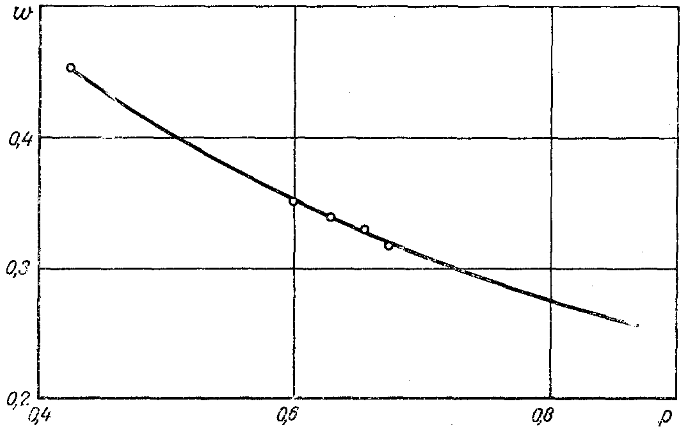

Fig. 2. Mean mass velocity was a function of the pressure p (the points denote the experiment described in [1], and the curve represents the approximation according to (16)).

For the scale which we have adopted

$$\left(\frac{\partial u}{\partial y}\right)_0 = \tau_0. \tag{2}$$

To determine $(\partial^2 u/\partial y^2)_0$ we will use the equation of motion in the Prandtl approximation, i.e.,

$$\rho u \frac{\partial u}{\partial x} + \widetilde{\mathrm{Re}}\,\rho v \frac{\partial u}{\partial y} = -\bar{k}\frac{dp}{dx} + \frac{1}{2r}\frac{\partial (r\tau)}{\partial y}, \tag{3}$$

where

$$\bar{k} = \frac{k-1}{2k}.$$

As $y \to 0$ we have

$$\left[\frac{\partial (r\tau)}{\partial y}\right]_0 = 2\bar{k}\frac{dp}{dx},$$

or

$$\left(\frac{\partial \tau}{\partial y}\right)_0 = \tau_0 + 2\bar{k}\frac{dp}{dx}. \tag{4}$$

In the usual manner, if we assume that

$$\tau = (\mu + \varepsilon)\frac{\partial u}{\partial y},$$

then

$$\left(\frac{\partial \tau}{\partial y}\right)_0 = (1 + \varepsilon_0)\left(\frac{\partial^2 u}{\partial y^2}\right)_0 + \left(\frac{\partial \mu}{\partial y} + \frac{\partial \varepsilon}{\partial y}\right)_0\left(\frac{\partial u}{\partial y}\right)_0.$$

Bearing in mind that $\varepsilon_0 = (\partial\varepsilon/\partial y)_0 = (\partial\mu/\partial y)_0 = 0$, we obtain

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_0 = \left(\frac{\partial \tau}{\partial y}\right)_0,$$

and from (4) we will therefore have

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_0 = \tau_0 + 2\bar{k}\frac{dp}{dx}. \tag{5}$$

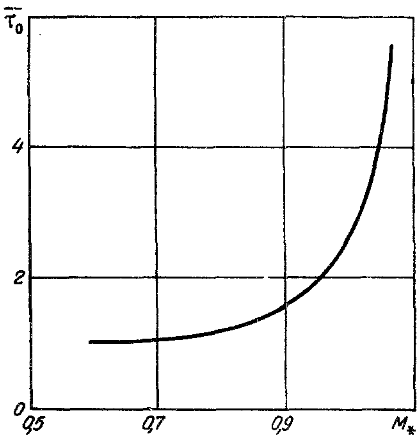

Fig. 3. Frictional stress $\bar{\tau}_0 = (\tau_0)/\tau_0|_{X=33}$ at the wall as a function of M_*.

Substituting (2) and (5) into (1), we find

$$r_{f0} = \frac{(1+\tau_0^2)^{3/2}}{\tau_0 + 2\bar{k}\,\dfrac{dp}{dx}}. \tag{6}$$

If we had $\tau_{0f} < \infty$, as $x \to x_f$, when $dp/dx \to -\infty$, it is not difficult from (6) to find that $r_{f0} \to 0$.

It is physically impossible to demonstrate such behavior of the longitudinal velocity profile, if we recall the results of the experiments described in [1]. We must therefore assume that $\tau_0 \to \infty$ as $x \to x_f$.

Having carried out the simple transformations, as $x \to x_f$ from (6) we obtain

$$r_{f0} = \frac{\left(1+\dfrac{1}{\tau_0^2}\right)^{3/2}}{\dfrac{1}{\tau_0^3} + 2\bar{k}\,\dfrac{dp}{dx}\Big/\tau_0^3} \to \frac{1}{2\bar{k}}\left(\frac{\tau_0^3}{dp/dx}\right)_f.$$

Hence it follows that as $x \to x_f$ the quantity τ_0 must approach infinity as $(-dp/dx)^{\beta}$, where $\beta \geq 1/3$. For $\beta = 1/3$ the curvature radius r_{f0} remains limited, while for $\beta > 1/3$ it increases without bound in absolute magnitude.

Near $x = x_f$ we can thus assume that

$$\tau_0 = A\left(-\frac{dp}{dx}\right)^{\beta} + \cdots, \tag{7}$$

where $\beta \geq 1/3$.

For the upper bound of β, we write the distribution of the tangential stress τ in the form

$$\tau = (1-y)\,[f + (\tau_0 - f)(1-y)^m], \tag{8}$$

where

$$f = -\left(\frac{\partial \tau}{\partial y}\right)_1, \quad m = -2\,\frac{\tau_0 + \bar{k}\,\dfrac{dp}{dx}}{\tau_0 - f}.$$

This distribution satisfies condition (4) at the wall, as well as the conditions $\tau|_{y=0} = \tau_0$, $\tau|_{y=1} = 0$.

In the limit case of the flow of an incompressible liquid $\tau_0 + \bar{k}(dp/dx) = 0$ and $m = 0$, and (8) changes into an ordinary linear relationship $\tau = \tau_0(1 - y)$.

The value of f can be determined from the equation of motion at the channel axis:

$$f = -\frac{dp}{dx}\left(\rho_1 u_1 \frac{du_1}{dp} + \bar{k}\right),$$

which is found from (3) as $r \to 0$.

Let us make the natural assumption that the quantity $\int_0^1 \tau dy$ remains limited at any cross section of the tube. Integrating (8), we find

$$\int_0^1 \tau dy = \frac{f}{2} + \frac{1}{2}\,\frac{\tau_0^2}{-\bar{k}\,\dfrac{dp}{dx}}\,\frac{\left(1-\dfrac{f}{\tau_0}\right)^2}{1+\dfrac{f}{\bar{k}\,\dfrac{dp}{dx}}}.$$

From this equation, on the strength of the boundedness of f, it follows that near $x = x_f$

$$\tau_0 = A\left(-\frac{dp}{dx}\right)^{\beta} + \cdots, \tag{9}$$

where $\beta \leq 1/2$.

In (7) and (9) we thus have

$$\frac{1}{3} \leqslant \beta \leqslant \frac{1}{2}. \tag{10}$$

The quantity τ_0 can be found from the equation of momentum

$$\tau_0 = -\frac{dp}{dx}\left(\bar{k} + G\frac{dw}{dp}\right), \tag{11}$$

where $w = (1/G)\int_0^1 \rho u^2(1-y)dy$ is the mean mass velocity,

Near $x = x_f$, from (9) and (11) we have

$$A = \left(-\frac{dp}{dx}\right)^{1-\beta}\left(\bar{k} + G\frac{dw}{dp}\right), \tag{12}$$

from which, on the strength of the boundedness of A and the condition $dp/dx \to -\infty$ as $x \to x_f$, bearing in mind that $\beta < 1$, we find

$$G\left(\frac{dw}{dp}\right)_f = -\bar{k}. \tag{13}$$

Near $x = x_f$ the pressure p can be presented as

$$p = p_f + a(X_f - X)^{\alpha} + \cdots, \tag{14}$$

where $\alpha < 1$.

It is not difficult to express α in terms of β, expanding the indeterminacy in (12) as $x \to x_f$.

In (12), bringing x to x_f, on the basis of the l'Hôpital rule, we find

$$A = \frac{\bar{k} + G\dfrac{dw}{dp}}{\left(-\dfrac{dp}{dx}\right)^{1-\beta}} \to \frac{G}{1-\beta}\frac{d^2w}{dp^2}\frac{\left(-\dfrac{dp}{dx}\right)^{3-\beta}}{-\dfrac{d^2p}{dx^2}}.$$

Assuming that $(d^2w/dp^2)_f \neq 0$, from this relationship and from (14) we find that

$$\alpha = \frac{1-\beta}{2-\beta},$$

whence, with (10), we find that

$$\frac{1}{3} \leqslant \alpha \leqslant \frac{2}{5}.$$

If we assume that $(d^2w/dp^2)_f = 0$, for the determination of A we need further expansion of the indeterminacy. If $(d^3w/dp^3)_f \neq 0$, we have $\alpha = (2-\beta)/(3-\beta)$, or with consideration of (10),

$$0.6 \leqslant \alpha \leqslant 0.625.$$

Processing the experimental data from [1], we became convinced that $\alpha < 0.6$.

We must therefore have $1/3 \leq \alpha \leq 2/5$ and $(d^2w/dp^2)_f \neq 0$. As the approximation relationship for p(X) near the exhaust we can take

$$p = p_f + a(X_f - X)^{\alpha} + b(X_f - X)^{\gamma}, \tag{15}$$

where $\gamma > \alpha$.

Figure 1 shows the function p(X) for the "B" experiment in [1]. Here we have $\alpha = 0.35$, $\gamma = 0.99808$, $a = 0.09619$, and $b = 0.00147$.

As we can see, approximation (15) shows good agreement with the experiment in a rather large region of X values before the exhaust. From (15) it is now easy to determine the derivative dp/dx.

To calculate τ_0 from (11) we also have to know the derivative dw/dp. Considering (13), we approximate the mean mass velocity w in a rather small area around the exhaust with the expression

$$w = w_f - \frac{\bar{k}}{G}(p - p_f) + a_2 (p - p_f)^2. \quad (16)$$

Here w_f is determined from the experimental values for $n_f = 1/15.4$, for the flow rate, and for p_f. We find the magnitude of a_2 by using the method of least squares for the values of w found from the experimental data in [1]. It proved to be equal to 0.4140.

Figure 2 shows the approximation curve (16) and the experimental values of w. The agreement is quite good.

Figure 3 shows $\bar{\tau}_0 = \tau_0/\tau_0|_{X=33}$, calculated on the basis of (11), (15), and (16), as a function of the Mach number M_*, determined from the one-dimensional model, i.e., from the equation

$$p \frac{w_*}{1 - w_*^2} = G. \quad (17)$$

The processing of the experiments described in [1] with consideration of the limit relationships thus leads to the conclusion that the surface friction τ_0 increases with an increase in M_*.

These results are in qualitative agreement with the experiment designed to measure surface friction directly at the wall of a tube [4].

NOTATION

x is the longitudinal coordinate;
y is the transverse coordinate;
r is the tube radius;
u is the longitudinal velocity component;
v is the transverse velocity component;
p is the pressure;
ρ is the density;
μ is the dynamic viscosity;
ε is the coefficient of turbulent viscosity;
k is the isentropic exponent;
G is the mass flow rate;
w is the mean mass velocity;
M is the Mach number;
Re is the Reynolds number;
τ is the tangential stress.

Symbols

0 denotes the wall of the tube;
1 denotes the tube axis;
f denotes the final section of the channel;
X = 33 is the cross section where the boundary layer fills the entire cross section of the tube [1];
θ is a quantity determined from the stagnation temperature;
* denotes a quantity determined on the basis of (17).

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